1. \( f(x, y, z) = 2x^2 + y^3 \) \( \Rightarrow \nabla f(x, y, z) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 4x, 3y^2, 0 \rangle \)
   \( \therefore \nabla f_{(1,1,2)} = \langle 4 \cdot 1, 3 \cdot 1^2, 0 \rangle = \langle 4, 3, 0 \rangle \)
   
   **The directional derivative of \( f \) at \((1,1,2)\) in the direction \( \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle \) is:**
   
   \[ \nabla f_{(1,1,2)} \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle = \langle 4, 3, 0 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle = \frac{4}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \frac{2 \cdot 5}{\sqrt{2}} = 2 \sqrt{5}; \]
   
   [Note that this could also be computed as \( \nabla f_{(1,1,2)} \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \rangle \).]

2. \( f(x, y) = x + 2xy - 3y^2 \Rightarrow \nabla f_{(x,y)} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle 1 + 2y, 2x - 6y \rangle \)
   \( \therefore \nabla f_{(1,2)} = \langle 1 + 2 \cdot 2, 2 \cdot 1 - 6 \cdot 2 \rangle = \langle 5, -10 \rangle \)
   
   **The directional derivative of \( f \) at \((1,2)\) in the direction \( \langle \frac{3}{5}, \frac{2}{5} \rangle \) is:**
   
   \[ \nabla f_{(1,2)} \cdot \langle \frac{3}{5}, \frac{2}{5} \rangle = \langle 5, -10 \rangle \cdot \langle \frac{3}{5}, \frac{2}{5} \rangle = 3 - 8 = -5; \]

3. *The only new fact in this problem is that directional derivatives are always computed FROM direction vectors, i.e., unit vectors. To compute a directional derivative in the direction of some arbitrary vector, first divide that vector by its length (in order to obtain a direction vector).*

(b) \( f(x, y, z) = e^x + y^2 \Rightarrow \nabla f_{(x,y,z)} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle e^x, 2y, 0 \rangle \)
   
   \( \therefore \nabla f_{(1,1,1)} = \langle e^1, 1, 0 \rangle = \langle e, 1, 0 \rangle \)
   
   **The direction of \( \langle 1, 1, 1 \rangle \) is \( \langle 1, 1, 1 \rangle \) \( \Rightarrow \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \)
   
   **Our directional derivative is:**
   
   \[ \nabla f_{(1,1,1)} \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle = \langle e, 1, 0 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle = \frac{e}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{e + 1}{\sqrt{3}}; \]

(b) \( f(x, y, z) = x^2 + 2y^2 + 3z^2 \) \( \Rightarrow \nabla f_{(x,y,z)} = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 2x, 4y, 6z \rangle \)
   
   \( \therefore \nabla f_{(1,2,3)} = \langle 2 \cdot 1, 4 \cdot 2, 6 \cdot 3 \rangle = \langle 2, 8, 3 \rangle \)

   **The tangent plane is thus given by:**
   
   \( (P - P_0) \cdot \vec{n} = 0 \) \( \Rightarrow (x, y, z) - (1, 2, 3) \cdot \langle 2, 8, 3 \rangle = 0 \)

\[ \begin{align*}
   & (x-1, y-2, z-3) \cdot \langle 2, 8, 3 \rangle = 0 \\
   & 2(x-1) + 8(y-2) + 3(z-3) = 0 \\
   & 3x + 8y + 3z = 20. 
\end{align*} \]
(6) **Plane Tangent to** \( y^2 - x^2 = 3 \) **at** \((1,1,8)\):

Let \( f(x,y,z) = y^2 - x^2 \); then this surface is the isocontour \( f^{-1}(3) \).

\[
\nabla f_{(x,y,z)} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} -2x \\ 2y \\ 0 \end{pmatrix}.
\]

\[
\therefore \quad \nabla f_{(1,1,8)} = \begin{pmatrix} -2 \cdot 1 \\ 2 \cdot 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}.
\]

The tangent plane is thus given by:

\[
(p - p_0) \cdot \mathbf{n} = 0 \quad \Rightarrow \quad ((x,y,z) - (1,1,8)) \cdot (-2,2,0) = 0.
\]

\[
\begin{align*}
    (x-1, y-2, z-8) \cdot (-2,2,0) &= 0 \\
    -2(x-1) + 2(y-2) &= 0 \\
    2x + 2y &= 6.
\end{align*}
\]

(8) **Plane Tangent to** \( z = x^2 + y^2 \) **at** \((1,1,3)\):

Let \( f(x,y,z) = z - x^2 - y^2 \); then this surface is the isocontour \( f^{-1}(3) \).

\[
\nabla f_{(x,y,z)} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}.
\]

\[
\therefore \quad \nabla f_{(1,1,3)} = \begin{pmatrix} -2 \cdot 1 \\ -2 \cdot 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}.
\]

The tangent plane is thus:

\[
(p - p_0) \cdot \mathbf{n} = 0 \quad \Rightarrow \quad ((x,y,z) - (1,1,3)) \cdot (-2,-2,1) = 0.
\]

\[
\begin{align*}
    (x-1, y-1, z-3) \cdot (-2,-2,1) &= 0 \\
    -2(x-1) - 2(y-1) + z &= 0 \\
    2x + 2y - z &= 6.
\end{align*}
\]

(9) Who cares if these surfaces are given as graphs? Rewrite them as isocontours, and it's the same game.

(a) **Plane Tangent to** \( z = x^3 + y^3 + 6xy \) **at** \((1,1,7)\):

\[
\begin{align*}
    z &= x^3 + y^3 + 6xy \quad \Rightarrow \quad z - x^3 - y^3 - 6xy &= 0, \text{ so this surface is an isocontour of } f(x,y,z) = z - x^3 - y^3 - 6xy.
\end{align*}
\]

\[
\nabla f_{(x,y,z)} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{18x^2}{3} + 6y \\ \frac{18y^2}{3} + 6x \\ 1 \end{pmatrix}.
\]

\[
\therefore \quad \nabla f_{(1,1,7)} = \begin{pmatrix} 18 \cdot 1^2 + 6 \cdot 1 \\ 18 \cdot 1^2 + 6 \cdot 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 24 \\ 24 \\ 1 \end{pmatrix}.
\]

The tangent plane is thus:

\[
(p - p_0) \cdot \mathbf{n} = 0 \quad \Rightarrow \quad ((x,y,z) - (1,1,7)) \cdot (24,24,1) = 0.
\]

\[
\begin{align*}
    (x-1, y-1, z-7) \cdot (24,24,1) &= 0 \\
    24(x-1) + 24(y-1) + z &= 0 \\
    24x + 24y - z &= 6.
\end{align*}
\]

(b) **Plane Tangent to** \( z = (\cos x)(\cos y) \) **at** \((0,\pi/2,0)\):

\[
\begin{align*}
    z &= (\cos x)(\cos y) \quad \Rightarrow \quad z - (\cos x)(\cos y) &= 0, \text{ so this surface is an isocontour of } f(x,y,z) = z - (\cos x)(\cos y).
\end{align*}
\]

\[
\nabla f_{(x,y,z)} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sin x \cos y \\ -\sin y \cos x \\ 1 \end{pmatrix}.
\]

\[
\therefore \quad \nabla f_{(0,\pi/2,0)} = \begin{pmatrix} -\sin 0 \cos \frac{\pi}{2} \\ -\sin \frac{\pi}{2} \cos 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.
\]

The tangent plane is thus:

\[
(p - p_0) \cdot \mathbf{n} = 0 \quad \Rightarrow \quad ((x,y,z) - (0,\pi/2,0)) \cdot (0,1,1) = 0.
\]

\[
\begin{align*}
    (x-0, y-\frac{\pi}{2}, z-0) \cdot (0,1,1) &= 0 \\
    y + z &= \frac{\pi}{2}.
\end{align*}
\]
13. UNIT NORMAL TO THE SURFACE \( \cos(xy) = e^z - 2 \) AT \((1, \pi, 0)\):

\[
\cos(xy) = e^z - 2 \implies \cos(xy) - e^z = -2, \text{ so this surface is an isosurface of } f(x,y,z) = \cos(xy) - e^z.
\]

\[
\nabla f(x,y,z) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \left\langle -y \sin(xy), -x \sin(xy), -e^z \right\rangle.
\]

\[
\therefore \nabla f(1, \pi, 0) = \left\langle -\pi \sin(1 \cdot \pi), -1 \sin(1 \cdot \pi), -e^0 \right\rangle = \left\langle 0, 0, -1 \right\rangle.
\]

(We normally would have to divide this by its length to obtain a unit vector, but it already is one.)

21. \( \mathbf{r} = \langle x, y, z \rangle \), \( \mathbf{v} = \| \mathbf{r} \| = \| \langle x, y, z \rangle \| = \sqrt{x^2 + y^2 + z^2} \)

\[
\nabla \left( \frac{1}{\mathbf{v}} \right) = \nabla \left( \frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \nabla \left( (x^2 + y^2 + z^2)^{-1/2} \right).
\]

\[
\frac{\partial}{\partial x} \left[ \left( x^2 + y^2 + z^2 \right)^{-1/2} \right] = -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} \cdot 2x = \frac{-x}{\sqrt{x^2 + y^2 + z^2}^3} = \frac{-x}{r^3}.
\]

By symmetry, \( \frac{\partial}{\partial y} = \frac{-y}{r^3} \) and \( \frac{\partial}{\partial z} = \frac{-z}{r^3} \)

So \( \nabla \left( \frac{1}{\mathbf{v}} \right) = \left\langle \frac{-x}{r^3}, \frac{-y}{r^3}, \frac{-z}{r^3} \right\rangle \)

\[
= -\frac{1}{r^3} \mathbf{r}
\]

\[
= -\frac{\mathbf{v}}{r^3} \checkmark
\]
31. A particle is ejected from the surface \( x^2 + y^2 - z^2 = 1 \) at the point \((1,1,\sqrt{3})\) along the normal pointed toward the xy-plane at time \( t = 0 \), with a speed of 10 units/sec. When + where does it cross the xy-plane?

Let’s break this down into two steps:

1. Find an expression giving the ejected particle’s path.
2. Determine when + where this crosses the xy-plane.

1. To determine a straight-line path of motion, we need a point + a velocity vector.

\[ \mathbf{P} = \mathbf{P}_0 + (t - t_0) \mathbf{V} \]

\[ \mathbf{P}_0 = (1,1,\sqrt{3}) \quad \text{(at time } t=0) \]

Velocity vector: Since we’re starting at \((1,1,\sqrt{3})\), the normal vector to the surface is \(\mathbf{N} = \nabla f = \langle 2x, 2y, -2z \rangle = \langle 2, 2, -2\sqrt{3} \rangle \)

\[ 2\mathbf{V}, (1,1,\sqrt{3}) = \langle 2, 2, -2\sqrt{3} \rangle \]

Two things to consider for \(\mathbf{V}\):

- **Pointed toward the xy-plane**:
  - Since we’re starting at \((1,1,\sqrt{3})\), the xy-plane will be “below” us, so we should move downward. i.e., the \(z\) component should be negative, which it is. \(\langle 2, 2, -2\sqrt{3} \rangle\)

- **Speed 10 units/sec**:
  - The magnitude of \(\mathbf{V}\) should be 10. As we currently have it,
    \[ \|\langle 2, 2, -2\sqrt{3} \rangle\| = \sqrt{4 + 4 + 12} = \sqrt{20} = 2\sqrt{5} \]
  - So, if we scale it by a factor of \(\frac{10}{2\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{10}}{2}\), we’ll be all set: \(\mathbf{V} = \frac{\sqrt{10}}{2} \langle 2, 2, -2\sqrt{3} \rangle\)

\[ \therefore \text{Our path is } \mathbf{P} = (1,1,\sqrt{3}) + (t-0) \frac{\sqrt{10}}{2} \langle 2, 2, -2\sqrt{3} \rangle \]

<table>
<thead>
<tr>
<th>(t)</th>
<th>(x)</th>
<th>(y)</th>
<th>(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>(\sqrt{3})</td>
</tr>
<tr>
<td>(t)</td>
<td>(1 + 2\sqrt{5})</td>
<td>(1 + 2\sqrt{5})</td>
<td>(3 - 2\sqrt{5})</td>
</tr>
</tbody>
</table>

2. To determine when + where this point crosses the xy-plane:

- Our point’s position at time \(t\) is given by:
  \[(x,y,z) = (1 + 2\sqrt{5}, 1 + 2\sqrt{5}, 3 - 2\sqrt{5})\]

- The xy-plane is given by \(z = 0\)...

So we can solve for \(t\) to determine the time of crossing:

\[ 0 = z = 3 - 2\sqrt{5} \]

\[ 2\sqrt{5} = 3 \]

\[ t = \frac{3}{2\sqrt{5}} = \frac{3}{2\sqrt{5}} = \frac{\sqrt{5}}{2} \text{, time of crossing} \]

At time \(t = \frac{\sqrt{5}}{2}\), we can find the point’s position via the path:

\[(x,y,z) = (1 + 2\sqrt{5}, 1 + 2\sqrt{5}, 3 - 2\sqrt{5}) \]

= \(1 + 2 \cdot \frac{\sqrt{5}}{2\sqrt{5}}, 1 + 2 \cdot \frac{\sqrt{5}}{2\sqrt{5}}, 3 - 2 \cdot \frac{\sqrt{5}}{2\sqrt{5}})\)

= \(1 + \frac{\sqrt{5}}{2}, 1 + \frac{\sqrt{5}}{2}, 3 - \frac{\sqrt{5}}{2}\)

\[= (1 + \sqrt{5}, 1 + \sqrt{5}, 0) \]

Point at which the particle crosses the xy-plane.