2. Define $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ by $F(x,y,z) = xy + z + 3x^2z$, and let $S$ be the isoset $F^{-1}([4]) = \{(x,y,z) : xy + z + 3x^2z = 4\}$.

Since $F$ is smooth, the implicit function theorem tells us that on $S$, we can solve for $z$ as a smooth function of $x,y$ near a point $p$ if $\frac{\partial F}{\partial z}(p) \neq 0$.

We see $\frac{\partial F}{\partial z} = 1 + 15x^2z$, so $\frac{\partial F}{\partial z}(1,0,1) = 16 \neq 0$. Thus there is a smooth function $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ solving for $z$ on $S$ as a function of $x,y$ near $(1,0,1)$.

In particular, setting $z = g(x,y)$ gives us $(x,y,z) \in S = F^{-1}([4])$, so $F(x,y,g(x,y))$ is constant at 4.

Applying the chain rule to the composition:

$$F(x,y,z) = \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$$

As matrices in coordinates,

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & 1 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

At $(1,0,1)$, $\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & 1 \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 6 \end{bmatrix}$, so the derivative at $(1,0,1)$ is:

$$\begin{bmatrix} 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 + 16 \frac{\partial g}{\partial x} & 1 + 16 \frac{\partial g}{\partial y} \end{bmatrix}$$

But the composition was constant, so this must be $\begin{bmatrix} 0 & 0 \end{bmatrix}$, which allows us to solve for $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$:

$$3 + 16 \frac{\partial g}{\partial x} = 0 \Rightarrow \frac{\partial g}{\partial x} = -\frac{3}{16}, \quad 1 + 16 \frac{\partial g}{\partial y} = \frac{\partial g}{\partial y} = -\frac{1}{16}.$$

7. Setting $F(x,y,z) = x^3z^2 - z^3yx$, we're looking at the isocontour $S = F^{-1}([0]) = \{(x,y,z) : x^3z^2 - z^3yx = 0\}$.

As in #2, we'll be able to solve for $z$ as a function of $x,y$ near a point $p$ if $\frac{\partial F}{\partial z}(p) \neq 0$.

$\frac{\partial F}{\partial z} = 2x^3z - 3z^2yx$, so $\frac{\partial F}{\partial z}(1,1,1) = 2 - 3 = -1 \neq 0$, and we can write the points of $S$ as $(x,y,g(x,y))$ near $(1,1,1)$, for some smooth $g: \mathbb{R}^2 \rightarrow \mathbb{R}$.

Just as in #2, we can set up the chain rule and solve for $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ at $(1,1)$, obtaining:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial g}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial g}{\partial z} = -\frac{1}{\frac{\partial F}{\partial z}}$$

But $\frac{\partial F}{\partial z} = 3x^2z^2 - 2yz$, so at $(1,1,1)$, $\frac{\partial F}{\partial z} = 3 - 1 = 2$, and thus $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial x} = -\frac{1}{2} - 1 = -\frac{3}{2}$.

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial g}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial g}{\partial z} = -\frac{1}{\frac{\partial F}{\partial z}}$$

But $\frac{\partial F}{\partial y} = -3x^2z$, so at $(1,1,1)$, $\frac{\partial F}{\partial y} = -1$, and thus $\frac{\partial g}{\partial y} = \frac{\partial g}{\partial y} = -\frac{1}{-1} \cdot -1 = -1$.

To show that we can't do this near $(0,0,0)$ we must work with the equation directly:

$$x^3z^2 - z^3yx = 0 \iff x^2(x^2-y^2) = 0 \iff x = 0 \text{ or } y = 0 \text{ or } x^2 = y^2.$$

$S$ contains all points $(0,y,z)$!

Thus when $(x,y) = (0,0)$, we have all points $(0,0,z)$ on $S$.

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Since there is more than one $z$ near $(0,0,0)$ for a single pair $(x,y) = (0,0)$, it can't be locally written as a function of $x,y$. 
9. If we set \( F: E^4 \rightarrow E^2 \), \( F(x,y,u,v) = (y + x + uv, uxy + v) \), then we're looking at the shape:

\[ S = F^{-1}(0,0) = \{(x,y,u,v) : y + x + uv = 0 = uxy + v\} \]

The implicit function tells us that we can locally solve for \((u,v)\) as a function of \((x,y)\) if the following matrix is nonsingular:

\[
\begin{bmatrix}
\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\
\frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}
\end{bmatrix} = \begin{bmatrix}
v & u \\
x & 1
\end{bmatrix}
\]

We can check this via the determinant:

\[ \text{det} \begin{bmatrix} u & v \\ x & 1 \end{bmatrix} = v - xuy \]

\[ \therefore \text{we can locally solve for } (u,v) \text{ in terms of } (x,y) \text{ near any point } (x,y,u,v) \text{ with } v - xuy \neq 0. \]

10. The inverse function theorem tells us that a smooth mapping \( (x,y,z) \mapsto (u,v,w) \) can be locally inverted to obtain a smooth function \( (x,y,z) \mapsto F^{-1}(u,v,w) \). If \( D F_p \) is an invertible linear transformation, we can check this by finding a matrix for \( D F_p \) and checking whether that matrix is nonsingular via the determinant:

\[
D F = \begin{bmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\
\frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\
\frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z}
\end{bmatrix} = \begin{bmatrix}
1+y & x & xy \\
y & 1+x & 0 \\
2 & 0 & 1+x+xy
\end{bmatrix}
\]

At \((x,y,z) = (0,0,0)\), this is:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \rightarrow \text{determinant} = 1
\]

\[ \therefore \text{nonsingular!} \]

\[ \therefore \text{we can locally solve this system for } x,y,z \text{ as a function of } u,v,w \text{ near the point } (0,0,0). \]