**M&T §4.4**

The operator \( \nabla \) = \( \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \) gives us a convenient means of computing the divergence (\( \nabla \cdot \mathbf{F} \)) and curl (\( \nabla \times \mathbf{F} \)) of a vector field \( \mathbf{F} \) on \( \mathbb{R}^3 \) or \( \mathbb{R}^2 \).

2. \( \nabla \times (xy, xz, yz) = \langle yz, xz, xy \rangle \)

   \[ \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (yz) + \frac{\partial}{\partial y} (x) + \frac{\partial}{\partial z} (xy) = 0 + 0 + 0 = 0 \]

   (incompressible)

3. \( \nabla \times (y, x + \cos x, x + e^y) \)

   \[ \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (y) + \frac{\partial}{\partial y} (x + \cos x) + \frac{\partial}{\partial z} (x + e^y) = 1 + 1 + 1 = 3 \]

   (uniformly expanding)

5. (ok, figure 4.4.11 is awful... but look at the lengths of the arrows and the sizes of the "boxes" they've drawn!)

   * Top-left + bottom-right quadrants:
     \( \nabla \cdot \mathbf{F} < 0 \), because the box is compressing in \( \nabla \)

   * Bottom-left + top-right quadrants:
     \( \nabla \cdot \mathbf{F} > 0 \), because the box is expanding in \( \nabla \)

13. \( \nabla \times (y, x, z) \)

   \[ \nabla \times \mathbf{F} = \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{vmatrix} = \langle 2, 0, 0 \rangle \]

   \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (x) + \frac{\partial}{\partial y} (y) + \frac{\partial}{\partial z} (z) = 1 + 1 + 1 = 3 \)

   (irrotational)

14. \( \nabla \times (y, x, z) \)

   \[ \nabla \times \mathbf{F} = \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{vmatrix} = \langle 2, 0, 0 \rangle \]

   \( \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x} (y) + \frac{\partial}{\partial y} (x) + \frac{\partial}{\partial z} (z) = 1 + 1 + 1 = 3 \)

   (irrotational)

* note that #2 + #15 combine to show a nonconstant field that's both incompressible and irrotational!

24. (a) \( \nabla \times (\nabla f) \) is a vector field

   (b) \( \nabla \cdot (\nabla f) \) is nonsense

   (c) \( \nabla \cdot (\nabla f) \) is a scalar field

   (d) \( \nabla \times (\nabla f) \) is nonsense

   (e) \( \nabla \cdot (\nabla f) \) is nonsense

   (f) \( \nabla \times (\nabla f) \) is nonsense

25. (a) \( \nabla \times (\nabla f) \) is nonsense

   (b) \( \nabla \cdot (\nabla f) \) is nonsense

   (c) \( \nabla \cdot (\nabla f) \) is nonsense

   (d) \( \nabla \times (\nabla f) \) is a vector field

   (e) \( \nabla \cdot (\nabla f) \) is nonsense

   (f) \( \nabla \times (\nabla f) \) is a scalar field

30. \( \int (x, y, z) = x^2 + xy + x^2 \)

   \[ \nabla \times (\nabla f) = \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{vmatrix} = \langle 1, 1, 1 \rangle = \langle 0, 0, 0 \rangle = 0 \]

   (gradient fields are irrotational!)

34. Because gradient fields are irrotational, \( \mathbf{F} \) can be a gradient field only if \( \nabla \cdot \mathbf{F} = 0 \).

   * Note that even if \( \nabla \times \mathbf{F} \neq 0 \), \( \mathbf{F} \) still might not be a gradient field — so this is only a negative test.

   IF \( \mathbf{F}(x, y, z) = \langle x^2 + y^2, -2xy, o \rangle \), THEN

   \[ \nabla \times \mathbf{F} = \begin{vmatrix} x & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{vmatrix} = \langle 2, -2, 0 \rangle \]

   (the first two coordinates of \( \nabla \times \mathbf{F} \) always come out to zero if \( \mathbf{F} \) is a vector field in the plane — why? )

   Because \( \nabla \times \mathbf{F} \) is not identically 0, \( \mathbf{F} \) cannot be a gradient field.
37. \( \mathbf{F}(x,y,z) = \langle 2x^2, 1, y^3 \rangle \), \\
\( \mathbb{f}(x,y,z) = x^2 y \)

(a) \( \mathbb{f} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle = \langle 2xy, x^2, 0 \rangle \)

(b) \( \partial \times \mathbb{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 & 1 & y^3 \end{vmatrix} = \langle 3y^2, 4x^2 - y^3, 0 \rangle \) \\
= \langle 3y^2 - x, 4x^2 - y^3, 0 \rangle

(c) \( \mathbb{F} \cdot \hat{i} = \langle 2x^2, 1, y^3 \rangle \cdot \langle 2xy, x^2, 0 \rangle \\
= \begin{vmatrix} i & j & k \\ 2x^2 & 1 & y^3 \\ 2xy & x^2 & 0 \end{vmatrix} = \langle 0 - x^3 y^2, 2x^2 y^2, 0 \rangle \) \\
= \langle -x^3 y^2, 2x^2 y^2, 0 \rangle

(d) \( \mathbb{F} \cdot \hat{\mathbb{f}} = \langle 2x^2, 1, y^3 \rangle \cdot \langle 2xy, x^2, 0 \rangle \\
= 4x^2 y^2 + x^2 + 0 \\
= 4x^2 y^2 + x^2 \)

40. \( \mathbf{F}(x,y,z) = \langle 3x^2 y, x^3 + y^3, 0 \rangle \)

(a) \( \text{Curl } \mathbb{F} = \nabla \times \mathbb{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 y & x^3 + y^3 & 0 \end{vmatrix} = \langle 0, 0, 3x^2 - 3x^2 \rangle = \langle 0, 0, 0 \rangle = 0 \)

(b) \text{We already know everything we need to set this up if we're careful: }

\text{Suppose that } \mathbb{F} = \nabla \mathbb{f} \text{ for some scalar field } \mathbb{f}(x,y,z). \\
\text{Then } \nabla \mathbb{f} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \text{ and } \nabla \mathbb{f} = \langle 3x^2 y, x^3 + y^3, 0 \rangle \\
\text{so } \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle = \langle 3x^2 y, x^3 + y^3, 0 \rangle

\text{\therefore } \frac{\partial}{\partial x} = 3x^2 y \Rightarrow \mathbb{f}(x,y,z) = \int 3x^2 y dx = x^3 y + C(y,z)

\text{The only touchy part here is that a "constant" in the antiderivative with respect to } x \text{ could very well involve } y \text{ and/or } z!

\text{\therefore } \frac{\partial}{\partial y} = x^3 + y^3 \Rightarrow \mathbb{f}(x,y,z) = \int x^3 + y^3 dy = x^3 y + \frac{1}{4} y^4 + D(x,z)

\text{\therefore } \frac{\partial}{\partial z} = 0 \Rightarrow \mathbb{f}(x,y,z) = \int 0 dz = 0 + E(x,y).

\text{Thus our } \mathbb{f} \text{ must fit: }

\begin{enumerate}
\item \( \mathbb{f}(x,y,z) = x^3 y + \boxed{y} \text{ plus } \boxed{\text{stuff}} \)
\item \( \mathbb{f}(x,y,z) = x^3 y + \frac{1}{4} y^4 + \boxed{x \text{ plus } \text{stuff}} \)
\item \( \mathbb{f}(x,y,z) = \boxed{x \text{ plus } \text{stuff}} \)
\end{enumerate}
WHAT'S \( f(x,y,z) \)?

1. INSIST THAT THERE MUST BE AN \( x^3y \), WHICH FITS IN WITH \( x \) "stuff".
2. INSISTS THAT THERE MUST BE A \( \frac{1}{4}y^4 \), WHICH FITS IN WITH \( \frac{1}{4}y \) "stuff".
3. ANYTHING ELSE MUST BE \( \begin{align*}
&1. \text{ A function of } y, \text{ only,} \\
&2. \text{ A function of } x \text{ and } y, \text{ only,}
\end{align*} \\
\text{and } \begin{align*}
&3. \text{ A function of } x, y, \text{ only.}
\end{align*}
\text{i.e. - anything else must be an honest constant.}

\[ f(x,y,z) = x^3y + \frac{1}{4}y^4 + C \]

i.e., \( f(x,y,z) \) has \( F \) as its gradient field.

(Yes, this could also be found via trial and error, and we can check that \( \nabla f = F \)).