3. If \( B = [0,1] \times [0,1] \times [0,1] \) and \( f(x,y,z) = x^2, \)

THEN \( \int_B f = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} x^2 \, dz \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{1} \left[ \int_{z=0}^{1} x^2 \, dz \right] \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{1} x^2 \, dy \, dx = \int_{x=0}^{1} x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_{0}^{1} = \frac{1}{3} \cdot 1 = \frac{1}{3} \)

5. If \( B = [0,2] \times [-1,1] \times [0,1] \) and \( f(x,y,z) = 2x + 3y + z, \)

THEN \( \int_B f = \int_{x=0}^{2} \int_{y=-1}^{1} \int_{z=0}^{1} (2x + 3y + z) \, dz \, dy \, dx = \int_{x=0}^{2} \int_{y=-1}^{1} \left[ \int_{z=0}^{1} (2x + 3y + z) \, dz \right] \, dy \, dx = \int_{x=0}^{2} \int_{y=-1}^{1} (2x + 3y) \, dy \, dx = \int_{x=0}^{2} \left[ \int_{y=-1}^{1} 2x \, dy + 3y \right]_{y=-1}^{1} \, dx = \int_{x=0}^{2} \left[ 2x(-1) + 3 \cdot 1 \right] \, dx = \int_{x=0}^{2} 4x + 1 \, dx = \left[ 2x^2 + x \right]_{x=0}^{2} = (2 \cdot 2^2 + 2) - (2^2 + 2) = 8 + 2 = 10 \)

When slicing 3-dimensional regions, we often slice \( x, y, z \) first and \( z = a \) last. In this case, we can break our work into two steps:

1. Find the "shadow" of the region in the xy-plane.
2. Determine, for each \((x, y)\) in the shadow, the bottom and top values of \( z \) in the region.

This gives us the slicing in \( x, y, \) and \( z \) gives the range for \( z \).

* Often, the xy-shadow is not fully described explicitly; in this case, we can check anywhere any surfaces intersect to find the required information.

E.g., if \( f(x, y) + z = g(x, y) \) are bounding surfaces, find where they intersect, i.e., where \( f(x, y) = g(x, y) \).

13. The volume of the solid \( R \) bounded by:

\[
\begin{align*}
x &= y, \\
y &= 0, \\
z &= 0, \\
and \ x + y + z &= 1
\end{align*}
\]

Let's start with the shadow of \( R \):

This is a triangular region in the xy-plane, and we can describe it via:

\[
\begin{align*}
x &= 0, \\
y &= 0, \\
z &= 0, \\
\text{and} \quad x + y + z &= 1
\end{align*}
\]

What we have left tells us what \( z \) does above each point in this triangle:

\( z = 0 \) is one bound and \( x + y + z = 0 \) gives the other: \( z = -x - y \). Of these, \( -x - y \) is lower, so \( z = -x - y \).

This is so far every point in the triangle!

Finally, volume of \( R = \int_{R} 1 \)

\[
= \int_{x=0}^{1} \int_{y=0}^{x} \left[ \int_{z=0}^{1} 1 \, dz \right] \, dy \, dx
\]

\[
= \int_{x=0}^{1} \int_{y=0}^{x} \left[ \frac{y}{2} \right] \, dy \, dx
\]

\[
= \int_{x=0}^{1} x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_{x=0}^{1} = \frac{1}{3} - 0 = \frac{1}{3}
\]

\[
= \int_{x=0}^{1} x^2 \, dx = \left[ \frac{1}{3} x^3 \right]_{x=0}^{1} = \frac{1}{3} - 0 = \frac{1}{3}
\]
16. \[ \int_0^1 \int_0^x \left( \int_{y=0}^1 y + x^2 \, dy \right) \, dx \]
\[ = \int_0^1 \left[ \int_{y=0}^1 y + \frac{1}{2} x^2 \right]_0^1 \, dy \, dx \]
\[ = \int_0^1 \left[ \frac{1}{2} y^2 + \frac{1}{2} x^2 \right]_0^1 \, dy \, dx \]
\[ = \int_0^1 \frac{1}{2} x^2 + \frac{1}{6} x^4 \, dx = \left[ \frac{1}{12} x^4 + \frac{1}{30} x^5 \right]_0^1 = \frac{1}{12} + \frac{1}{30} = \frac{7}{60} \]

25. ("Shadow" Method: no x, y bounds are explicitly given, so they must be implicit in \( \sqrt[4]{x^2+4y^2} \geq 1 \))

The region will end when the bottom surface \( z = \sqrt[4]{x^2+4y^2} \) and the top surface \( z = 1 \) meet:

\[ \sqrt[4]{x^2+4y^2} = 1 \Rightarrow x^2+4y^2 = 1, 1 \leq y \leq \]

The unit circle.

\[ \begin{array}{c}
\text{The Shadow of } \mathcal{W} \text{ is:} \\
y = \pm \sqrt{1-x^2}
\end{array} \]

Solving for \( y \) in terms of \( x \),

\[ y = \pm \sqrt{1-x^2} \]

gives us the \( y \) range for each \( x \)-slice, so the shadow can be described as:

\[ \begin{cases} x: -1 \ldots 1 \\ y: -\sqrt{1-x^2} \ldots \sqrt{1-x^2} \end{cases} \]

The \( z \)-range was already given as \( z: \sqrt{x^2+y^2} \ldots 1 \), so

we can compute \( \int_{\mathcal{W}} f \) as

\[ \int_{x=-1}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^1 f(x,y,z) \, dz \, dy \, dx \]

28. (They're being tricky here — even though \( x + y \) have been completely described via \( x \leq 1, |y| \leq 1 \), the bottom surface \( z = 0 \) and the top surface \( x^2 + y^2 + z^2 = 1 \) cut off an even smaller region, so they affect the shadow.)

\[ z = 0 \text{ and } x^2 + y^2 + z^2 = 1 \]
\[ \Rightarrow x^2 + y^2 = 1 \]

As in #25, we can describe the unit disk by

\[ \begin{cases} x: -1 \ldots 1 \\ y: -\sqrt{1-x^2} \ldots \sqrt{1-x^2} \end{cases} \]

So, \( \int_{\mathcal{W}} f \) can be computed as:

\[ \int_{x=-1}^1 \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} f(x,y,z) \, dz \, dy \, dx \]