M&T §6.4

To set up an improper integral, first determine what about the shape is causing problems, then devise a means of expressing your shape as a limit of shapes that avoid the problem areas. Compute the integral over the "nice" shapes, then take the appropriate limit.

1. Here \( D = [0,1] \times [0,1] \) and \( f(x,y) = \frac{1}{\sqrt{xy}} \); we're asked to find \( \int_D f \).

Trouble spots are when \( x \) or \( y \) is zero... sketch \( D \) and set up a limit of regions that fill up \( D \) but avoid the \( x \)-\( y \) axes.

Let \( D_\varepsilon = [\varepsilon,1] \times [\varepsilon,1] \) for \( \varepsilon > 0 \). As \( \varepsilon \to 0^+ \), \( D_\varepsilon \) fills up more and more of \( D \), so we set

\[
\int_D f = \lim_{\varepsilon \to 0^+} \int_{D_\varepsilon} f.
\]

Now, \( \int_{D_\varepsilon} f = \int_{x=\varepsilon}^{1-\varepsilon} \int_{y=\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{xy}} \, dy \, dx \)

\[
= \left[ \frac{1}{\sqrt{x}} \right]_{x=\varepsilon}^{1-\varepsilon} \cdot \int_{y=\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{y}} \, dy
\]

\[
= [2\sqrt{1-\varepsilon} - 2\sqrt{\varepsilon}] \cdot \left[ 2\sqrt{1-\varepsilon} - 2\sqrt{\varepsilon} \right]^2
\]

\[
= (2 - 2\varepsilon)^2
\]

\[ \int_D f = \lim_{\varepsilon \to 0^+} \int_{D_\varepsilon} f = \lim_{\varepsilon \to 0^+} (2 - 2\varepsilon)^2 = 4, \quad \text{converges!} \]

10. \( \int_{x=0}^{a} \int_{y=0}^{a} \frac{x}{\sqrt{a^2-y^2}} \, dy \, dx \):

Region \( R \) is \( [0,1] \times [0,a] \)

Scalar field is \( f(x,y) = \frac{x}{\sqrt{a^2-y^2}} \).

Trouble when \( y = a \), so avoid it.

Set this up as a limit of integrals over regions \( R_t = [0,1] \times [0,t] \)

and take the limit as \( t \to a^- \):

\[
\int_{R_t} f = \lim_{t \to a^-} \int_{R_t} f = \lim_{t \to a^-} \int_{y=0}^{a} \int_{x=0}^{1} \frac{x}{\sqrt{a^2-y^2}} \, dx \, dy
\]

\[
= \left[ \int_{x=0}^{1} x \, dx \right] \left[ \int_{y=0}^{a} \frac{1}{\sqrt{a^2-y^2}} \, dy \right]
\]

\[
= \left[ \frac{1}{2} t^2 \right]_{x=0}^{1} \cdot \left[ -\frac{1}{a} \arcsin \frac{y}{a} \right]_{y=0}^{a}
\]

\[
= \left[ \frac{1}{2} - 0 \right] \cdot \left[ -\frac{1}{a} \arcsin \frac{a}{a} - 0 \right]
\]

\[
= \frac{1}{2a} \arcsin \frac{a}{a}
\]

\[
\therefore \int_{R_t} f = \lim_{t \to a^-} \int_{R_t} f = \lim_{t \to a^-} \int_{y=0}^{a} \int_{x=0}^{1} \frac{x}{\sqrt{a^2-y^2}} \, dx \, dy
\]

\[
= \frac{1}{2a} \cdot \frac{\pi}{2} = \frac{\pi}{4a}
\]