2. \[ \text{If } f(x, y, z) = 2 + 6 \text{ and } S \text{ is parameterized by } \Phi(u,v) = \left( u, \frac{y}{2}, z \right), \]

where \( u \leq 1, v \leq 0, z \leq 3 \), then \( \int_S f = \int_T f \cdot \mathbf{n} \), where \( T_u = (0,1,0) \) and \( T_v = (0,0,1) \), so

\[ \Phi_u = \left( \frac{y}{2}, 1, 0 \right), \quad \Phi_v = (0,0,1) \]

\[ \Phi_u \times \Phi_v = \left| \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right| = (0, -1, 1) \]

\[ \text{and the magnitude of } \Phi_u \times \Phi_v = \sqrt{1 + 1} = \sqrt{2} \]

\[ \int_S f = \int_T f \cdot \mathbf{n} = \sqrt{2} \int_0^1 \int_0^y u + z \, du \, dv = \frac{\sqrt{2}}{2} \int_0^1 \int_0^y u + 3 \, du \, dv = \frac{\sqrt{2}}{2} \int_0^1 \left( \frac{1}{2} y^2 + 6 ight) \, dv = \frac{1}{2} \int_0^1 \left( \frac{1}{2} y^2 + 6 ight) \, dv = \frac{1}{2} \frac{1}{2} = \frac{1}{4} \]

3. \[ \text{If } f(x, y, z) = x + y \]

and \( S \) is the triangle with vertices \((1,0,0), (0,2,0), (0,1,1)\), then \( \int_S f = \int_R f \cdot \mathbf{n} \), where \( R \) is the shadow of \( S \) in the xy-plane.

4. \[ \Phi(u,v) = (u, v, u, v) \]

5. (a) To show that \( \Phi \) 's image lies on \( z = x^2 - y^2 \), just compute:

\[ 4z = 4uv \quad x^2 - y^2 = (u + v)^2 - (u - v)^2 = (u^2 + 2uv + v^2) - (u^2 - 2uv + v^2) = 4uv \]

(b) If we integrate over the shadow \( R \) of this piece of the graph \( z = \frac{1}{2}(x^2 - y^2) \), we'll pick up a magnification factor of \( 9M = \left\| \Phi_u \times \Phi_v \right\| = \left\| \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 0 \end{array} \right\| = \sqrt{1 + \frac{1}{2} x^2 + \frac{1}{2} y^2} \]

So, we're down to \( \int_S f \cdot \mathbf{n} \, dA \), but this is an odd function of \( x \) - integrated over a region symmetric in \( x \), so it must balance out to 0.
10. \( \int_{S^3} x + y + z \), where \( S \) is the unit sphere in \( \mathbb{E}^3 \)

**Shortcut:** \( \int_{S} x = 0 \), because the shape \( S \) is symmetric in \( x \)

And each value of this scalar field on the \( x > 0 \) hemisphere is balanced by an opposite value on the (identical) \( x < 0 \) hemisphere, cancelling each other in the integral.

Similarly, \( \int_{S} y = 0 \) and \( \int_{S} z = 0 \), so adding,

\[ \int_{S} x + y + z = 0 \] as well.

In terms of pullbacks, we can justify this reasoning as follows: Take, e.g., \( \tilde{f}(x, y, z) = z \), and let \( \tilde{S} \) be the unit sphere.

If we define \( \tilde{S} \to S \) by \( \tilde{S}(x, y, z) \mapsto (x, y, -z) \)

(i.e., \( \tilde{S} \) mirrors \( S \) plane vertically),

then clearly \( M = 1 \) for \( \tilde{S} \), because it doesn’t stretch or shrink the shape \( S \). Well, then:

\[ \int_{S} f = \int_{S} (f \circ \tilde{S}) \cdot M = \int_{\tilde{S}} f \circ \tilde{S} \]

**This is the "R", but \( \tilde{S} \to S \) here**

But \( \int_{S} f = \int_{\tilde{S}} z \), while \( \int_{S} f \circ \tilde{S} = \int_{\tilde{S}} z = -\int_{\tilde{S}} z = -\int_{S} f \)

(Left-hand side) (Right-hand side)

Putting this together, \( \int_{S} f = -\int_{S} f \), so \( \int_{S} f = 0 \).
For practice, we'll set it up anyway:

Parametrize \( S \) in spherical coordinates \( \{ \phi : 0 \ldots \pi, \theta : 0 \ldots 2\pi \} \to \mathbb{R} \mathbb{E} \mathbb{R} \)

The mapping is, as usual, \( \Phi(\theta, \phi) = (x, y, z) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \), and as we've already computed at least a few times, \( M = |\text{swp}| \).

Setting \( f(x, y, z) = x + y + z \),

\[
\int_S f = \int_R (f \circ \Phi) \cdot M
\]

\[
= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} (\cos \theta \sin \phi + \sin \theta \sin \phi + \cos \phi) \sin \phi \, d\phi \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} \cos \theta \sin^2 \phi \, d\phi + \sin \theta \sin \phi \cos \phi \, d\phi \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \left[ \frac{\cos \theta \sin^2 \phi}{2} - \frac{\sin \theta \cos \phi}{\cos \phi} \right]_{\phi=0}^{\pi} \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \left( \frac{\cos \theta}{2} - \frac{\sin \theta}{\cos \phi} \right) \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \frac{\cos \theta}{2} \, d\theta - \int_{\theta=0}^{2\pi} \sin \theta \, d\theta
\]

\[
= \frac{1}{2} \left[ \sin \theta \right]_{\theta=0}^{2\pi} \quad \text{and} \quad \left[ -\cos \theta \right]_{\theta=0}^{2\pi}
\]

\[
= 0 - 0
\]

Similarly for the others...

17. \( S^2 \): Sphere of radius \( R \)

(a) \( \int_S x^2 = \int_S y^2 = \int_S z^2 \) because if we transform \( \mathbb{E}^3 \) accordingly (e.g., to switch the positions of the \( x \)- and \( y \)-axes), then the shape \( S \) (being a sphere) stays the same, but the field changes (e.g., from \( x^2 \) to \( y^2 \)).

In terms of pullbacks, let \( \Phi : S \to S \) be the mapping \( \Phi : (x, y, z) \mapsto (y, x, z) \), switching the \( x \)- and \( y \)-coordinates. It is clear that \( M = 1 \) for \( \Phi \), because \( \Phi \) doesn't stretch the sphere at all. If we let \( f(x, y, z) = x^2 \), then we know:

\[
\int_S f = \int_R (f \circ \Phi) \cdot M = \int_R f \circ \Phi (\text{since } M = 1)
\]

But \( \int_S f = \int_S x^2 \), and \( \int_S x^2 \circ \Phi = \int_S y^2 \).

This is the "\( R^2 \)" bot \( \Phi : S 
\rightarrow S \) here.

But \( \int_S f = \int_S x^2 \), and \( \int_S f \circ \Phi = \int_S y^2 \).

Similarly for the others...

(b) The cleverness lies in one observation:

\[
\int_S x^2 = \int_S y^2 = \int_S z^2,
\]

so \( \int_S x^2 + y^2 + z^2 = 3 \int_S x^2 \).

But on \( S \) (a sphere of radius \( R \)),

\[
x^2 + y^2 + z^2 = R^2 \quad \text{(constant!)}
\]

\[
\therefore \int_S x^2 + y^2 + z^2 = \int_S R^2 = |S| \cdot R^2 = 4\pi R^2.
\]

\[
\therefore \int_S x^2 = \frac{1}{3} \int_S x^2 + y^2 + z^2 = \frac{1}{3} (4\pi R^2) = \frac{4\pi}{3} R^2
\]