The derivative and scalar fields

Suppose that we have some smooth scalar field \( f : \mathbb{R}^n \to \mathbb{R} \), with its derivative \( Df \).

- From linear algebra, we know that the value of \( Df_p(\vec{v}) \) is always given by dotting \( \vec{v} \) with some vector, denoted by \( \nabla f_p \) or \( \text{grad}_p f \), called the gradient of \( f \) at \( p \); i.e.: \( Df_p(\vec{v}) = \nabla f_p \cdot \vec{v} \).

- Given a point \( p \in \mathbb{R}^n \) and a direction \( \vec{u} \in \mathbb{R}^n \), the directional derivative of \( f \) at \( p \) in the direction \( \vec{u} \) is the (scalar) rate of change of \( f \) when moving in direction \( \vec{u} \) from \( p \), i.e., \( Df_p(\vec{u}) \). Restating this in terms of the gradient, this rate of change is given by \( \nabla f_p \cdot \vec{u} \).

Interpreting this dot product geometrically as \( \|\nabla f_p\| \cos \theta \), we find that at the point \( p \):

- \( f \) increases most rapidly in the direction of \( \nabla f_p \), with \( \|\nabla f_p\| \) giving the rate.
- \( f \) decreases most rapidly in the direction opposite \( \nabla f_p \).
- \( f \) remains constant in directions \( \perp \nabla f_p \).

- Considering that \( p \mapsto \nabla f_p \) assigns to each point a vector, we see that \( \nabla f \) defines a vector field, called the gradient field of \( f \).

  - This vector field always points in the direction of most rapid increase for the function \( f \), and it is perpendicular to all isosets of \( f \).
The derivative and vector fields

- The **divergence** of a vector field $\vec{F}$ is a scalar field, denoted by $\text{div} \, \vec{F}$ or $\nabla \cdot \vec{F}$. This scalar field measures, at each point, how much the vector field pushes away ("diverges") from that point. A vector field for which $\text{div} \, \vec{F} \equiv 0$ is called **incompressible**.

- The **curl** of a vector field $\vec{F}$ is another vector field, denoted by $\text{curl} \, \vec{F}$ or $\nabla \times \vec{F}$. This vector field measures, at each point, how the vector field rotates ("curls") at that point. Specifically, the vector field exhibits right-handed rotation perpendicular to $\text{curl} \, \vec{F}$, with $\|\text{curl} \, \vec{F}\|$ giving the rate of rotation. A vector field for which $\text{curl} \, \vec{F} \equiv \vec{0}$ is called **irrotational**.