Integration in Euclidean space

In general, an integral represents a product in which one of the factors is not constant.

More specifically: suppose that a quantity we’re interested in computing is a product of the size (length, area, volume, time) of a shape and some scalar value, when the scalar value is constant. Then an integral is how we compute that same quantity when the value varies over the shape, e.g.:

- If $I$ is an interval and $h$ gives height, then $\text{Area} = |I|h$ when $h$ is constant. Thus when $h$ varies, $\text{Area} = \int_I h$.

- If $R$ is a 2-region and $h$ gives height, then $\text{Volume} = |R|h$ when $h$ is constant. Thus when $h$ varies, $\text{Volume} = \int_R h$.

- If $R$ is some shape and $\sigma$ gives density, then $\text{Mass} = |R|\sigma$ when $\sigma$ is constant. Thus when $\sigma$ varies, $\text{Mass} = \int_R \sigma$.

- If $\vec{F}$ is a velocity field on a curve $C$, choosing a unit tangent vector $\vec{T}$, Circulation $= |C| (\vec{F} \cdot \vec{T})$ when $\vec{F}, \vec{T}$ are constant. So when $\vec{F}$ and/or $\vec{T}$ vary, Circulation $= \int_C \vec{F} \cdot \vec{T}$; (if $\vec{F}$ is a force field, this quantity gives work).

- If $\vec{F}$ is a vector field on a surface $S$, choosing a unit normal vector $\vec{n}$, Flux $= |S| (\vec{F} \cdot \vec{n})$ when $\vec{F}, \vec{n}$ are constant. So when $\vec{F}$ and/or $\vec{n}$ vary, Flux $= \int_S \vec{F} \cdot \vec{n}$.
Methods of integration

We’ll be interested in computing integrals over three classes of shapes:

**Intervals:** \( \int_I f \) can be computed, just as in single-variable calculus, by the FTC:
\[
\int_a^b f = F(b) - F(a), \quad \text{where } F' = f.
\]

**Simple regions:** Integrals over simple regions can be computed via *iterated integration*, iteratively computing single-variable integrals as above.

**All other shapes:** [Use parametrization and pull-back]

To compute an integral \( \int_S f \) of some scalar field \( f \) over a complicated shape \( S \), we’ll:

- **Parametrize** \( S \) via a mapping \( \Phi : R \to S \), where \( R \) is a “simple” region or interval.
- **Pull back** \( f \) from \( S \) to \( R \), i.e., take the scalar field \( f \circ \Phi \) on \( R \).
- **Determine the magnification** \( \mathcal{M} \) of the mapping \( \Phi \).

Because the values of the scalar field \( f \circ \Phi \) correspond to those of \( f \), the only difference between integrating one or the other is the local magnification \( \mathcal{M} \) of the mapping.

Accounting for this, we can compute an integral over a complicated shape \( S \) by translating the problem to the simple region \( R \):

\[
\int_R (f \circ \Phi) \cdot \mathcal{M} = \int_S f
\]